KSU CET UNIT

FIRST YEAR NOTES

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23/9/2019

MODULE - IV

INFINITE SERIES

Sequence:

It is a function from $N \to \mathbb{R}$, $f: N \to \mathbb{R}$

e.g., $\left\{\frac{1}{n}\right\}_{n \in N} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$

Limit of a sequence:

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

Divergent sequence:

A sequence which has no limit is called a divergent sequence.

e.g., $1-1+1-1+1-1+ \cdots \infty$

Convergent sequence:

A sequence which has a limit is called a convergent sequence. It converges to the limit.

Infinite series:

An expression of the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots \infty$$

where $u_1, u_2, \ldots$ are called terms of series.

Sequence of Partial Sums:

e.g., $\frac{1}{3} = 0.333\ldots = 0.3 + 0.03 + 0.003 + \cdots$

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n}\right)$$

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \cdots + \frac{3}{10^n}$$

$$\frac{1}{10} S_n = \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n} + \frac{3}{10^{n+1}}$$

$$\left(1 - \frac{\cancel{9}1}{10}\right) S_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

$$\frac{9}{10} S_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$

$$\Rightarrow S_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$$

$$\lim_{n\to\infty} S_n = \frac{1}{3}\lim_{n\to\infty}\left(1 - \frac{1}{10^n}\right)$$

$$= \frac{1}{3}$$

$S_1 = \frac{3}{10} = 0.3$

$S_2 = \frac{3}{10} + \frac{3}{10^2} = 0.33$

$S_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3}$

$= 0.333$

Note: A ~~sequence~~ series is said to be convergent, if sequence of partial sums is convergent.

26/9/2019 Geometric Series

An expression of the form $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \cdots + ar^k + \cdots \infty$, where r is common ratio.

- If $|r| < 1$, the geometric series is convergent.
- If $|r| \geq 1$, the geometric series is divergent.

e.g., $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \infty$ is convergent as $|r| = \frac{1}{2} < 1$.

$1 - 1 + 1 - 1 + \cdots \infty$ is divergent as $|r| = 1$

Sum of convergent geometric series is given by

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

e.g., $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$a = 1$

$r = \frac{1}{2}$

$$\therefore \sum_{k=0}^{\infty} ar^k = \frac{1}{1-\frac{1}{2}} = 2$$

1. $1 + 2 + 4 + 8 + \cdots + \infty$.

Here, $a = 1$, $r = 2$

$|r| > 1$

$\therefore$ The given series is divergent.

2. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$

$$a = \frac{1}{2}, \quad r = \frac{-1}{2}$$

$$|r| < 1$$

$\therefore$ Series is convergent.

$$S = \frac{a}{1-r} = \frac{1/2}{1+1/2} = \underline{\underline{\frac{1}{3}}}$$

3. $1+1+1+ \cdots \infty$

$$a = 1$$
$$r = 1$$
$$|r| = 1$$

$\therefore$ series $\underline{\underline{\text{is divergent}}}$.

4. $1-1+1-1+ \cdots \infty$

$$a = 1$$
$$r = -1$$
$$|r| = 1$$

$\therefore$ Series $\underline{\underline{\text{is divergent}}}$.

5. $1 + x + x^2 + x^3 + \cdots \infty$

$$a = 1$$
$$r = x$$
$$|r| = |x|$$

If $|x|\ \cancel{x} < 1$, then the series is convergent.

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6. $\sum_{k=0}^{\infty} \frac{5}{4^k}$

$$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5 + \frac{5}{4} + \frac{5}{4^2} + \cdots \infty$$

$$a = 5$$

$$r = \frac{1}{4}$$

$$|r| < 1$$

$\therefore$ Series is convergent.

$$S = \frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \underline{\underline{\frac{20}{3}}}$$

7. $\sum_{k=1}^{\infty} 3^{2k} \cdot 5^{1-k} = \sum_{k=1}^{\infty} 9 \cdot \left(\frac{9}{5}\right)^{k-1}$

$$\sum_{k=1}^{\infty} 3^{2k} \cdot 5^{1-k} = 3^2 + \frac{3^4}{5} + \frac{3^6}{5^2} + \cdots$$

$$a = 9$$

$$r = \frac{9}{5}$$

$$|r| > 1$$

$\therefore$ series is divergent.

8. Find the rational number represented by the repeating decimal 0.784784784.......

$0.784784784\ldots = 0.784 + 0.000784 + 0.000000784 + \ldots$

Geometric series: $a = 0.784$
$r = 0.001$

So the given decimal is a sum of the geometric series with $a = 0.784$, $r = 0.001$.

Thus, $0.784784784\ldots$

$$= S_n = \frac{a}{1-r} = \frac{0.784}{1-0.001} = \frac{784}{999}$$

Q: Find all values of x for which the series ~~converges~~

(i) $\sum_{k=0}^{\infty} x^k$ (ii) $3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \cdots + \frac{3(-1)^k x^k}{2^k}$

converges.

(i) $\sum_{k=0}^{\infty} x^k$

$= 1 + x + x^2 + \cdots$

Geometric series with $a = 1$
$r = x$

It converges if $|x| < 1$.

When $|x| < 1$, its sum is $S = \frac{1}{1-x}$.

(ii) $3 - \frac{3x}{2} + \frac{3x^2}{4} - \frac{3x^3}{8} + \cdots + \frac{3(-1)^k x^k}{2^k}$

$a = 3$, $r = \frac{-x}{2}$

It converges if $|r|<1$

$$\Rightarrow \left|\frac{-x}{2}\right|<1$$

$$\Rightarrow \left|\frac{x}{2}\right|<1$$

$$\Rightarrow |x|<2$$

~~So~~ when the series converges.

$$\text{sum}, S = \frac{3}{1-\frac{-x}{2}} = \frac{6}{2+x}$$

Telescoping Sums

Q: Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots \infty$ converges or diverges. If it converges, find the sum.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots \infty$$

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1 + \left(\frac{-1}{2}+\frac{1}{2}\right) + \left(\frac{-1}{3}+\frac{1}{3}\right) + \cdots + \left(\frac{-1}{n}+\frac{1}{n}\right) - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

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$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)$$

$$= 1 - \lim_{n\to\infty}\frac{1}{n+1}$$

$$= \underline{\underline{1}}$$ Thus, we have Sn(sequence of partial sum) is convergent with limit 1.

(∴ S_n converges to 1)

∴ the series is convergent.

∴ sum of series, $S = \sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \underline{\underline{1}}$

Harmonic series

The one of the most important of all diverging series is the harmonic series,

$$\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+ \cdots \infty$$

p-Series

A p-series is an infinite series of the form

$$\sum_{k=1}^{\infty}\frac{1}{k^p} = \frac{1}{1}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$$

where, $p>0$.

Examples of p-series are:

1. $\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\cdots$

$\boxed{p=1}$

2. $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

$\boxed{p=2}$

3. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$

$\boxed{p=\frac{1}{2}}$

Theorem:

Convergence of p-series

The p-series $\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$

converges if $p>1$ and diverges if $0<p\leq 1$.

Using the theorem, $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent,

$\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent.

Q: Determine whether the following series converges or not:

1. $\sum k^{-2/3} = \sum \frac{1}{k^{2/3}} \Rightarrow p = \frac{2}{3} < 1 \Rightarrow$ divergent
2. $\sum \frac{1}{\sqrt[3]{k^5}} = \sum \frac{1}{k^{5/3}} \Rightarrow p = \frac{5}{3} > 1 \Rightarrow$ convergent
3. $\sum \frac{1}{k^e} \Rightarrow p = e = 2.7 > 1 \Rightarrow$ convergent.

3/10/2019

Test for Convergence

k^{th}-term test:

If $\sum_{k=1}^{\infty} a_k$ is convergent, then, $\lim_{k\to\infty} a_k = 0$.

Note: But the converse ~~is~~ need not be true.

e.g., $\sum_{k=1}^{\infty} \frac{1}{k}$, which is the harmonic series which is divergent.

k^{th} term is $\frac{1}{k}$

& $\lim_{k\to\infty} \frac{1}{k} = 0$

But $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

Note: • If $\lim_{k\to\infty} a_k \neq 0$, then the series will be divergent.

Q: Test the convergence of the series $\sum_{k=1}^{\infty} \frac{k+1}{k}$.

$$\sum_{k=1}^{\infty} \frac{k+1}{k} = \sum_{k=1}^{\infty} 1+\frac{1}{k}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} 1+\frac{1}{k} = 1 \neq 0$$

∴ The given series is divergent.

Q: Determine whether the series converges or diverges: (i) $\sum_{k=1}^{\infty} \frac{k}{k+1}$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k}{k+1}$$

$$= 1 \neq 0$$

∴ The series is divergent.

(ii) $\sum_{k=1}^{\infty} \sin k\pi$

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$$\sum_{k=1}^{\infty} \sin k\pi$$

Here, the k^{th} term of the series is

$a_k = \sin k\pi = 0$ for every k.

$\therefore$ the k-th term test cannot be used.

$\therefore$ The sequence of partial sum method is used.

$$S_1 = \sin\pi = 0$$

$$S_2 = \sin\pi + \sin 2\pi = 0$$

$$\vdots$$

$$S_n = a_1 + a_2 + \cdots + a_n = 0$$

Thus, we have, $\lim S_n = 0 = S$

$$S = \sum_{k=1}^{\infty} \sin k\pi = 0$$

$\therefore$ The series is convergent.

(iii) $\displaystyle\sum_{k=1}^{\infty} \frac{k^2+k+3}{3k^2+1}$

$$= \sum_{k=1}^{\infty} \frac{k^2\left(1+\frac{1}{k}+\frac{3}{k^2}\right)}{k^2\left(3+\frac{1}{k^2}\right)}.$$

$$= \sum_{k=1}^{\infty} \frac{1+\frac{1}{k}+\frac{3}{k^2}}{3+\frac{1}{k^2}}$$

$$\lim_{k\to\infty} \sum_{k=1}^{\infty} \frac{1+\frac{1}{k}+\frac{3}{k^2}}{3+\frac{1}{k^2}} = \frac{1}{3} \neq 0$$

$\therefore$ The series is divergent.

(iv) $\sum_{k=1}^{\infty} \left(1+\frac{3}{k}\right)^k$

$$\lim_{k\to\infty} \left(1+\frac{3}{k}\right)^k = e^3 \neq 0 \qquad \lim_{k\to\infty} \left(1+\frac{x}{k}\right)^k = e^x$$

$\therefore$ The series is divergent.

Comparison Test

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with non-negative terms and suppose that

$a_1 \leq b_1, a_2 \leq b_2, \dots a_k \leq b_k$ $\left(\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k\right)$.

(a) If the 'bigger' series $\sum_{k=1}^{\infty} b_k$ converges, then the 'smaller' series $\sum_{k=1}^{\infty} a_k$ also converges.

(b) If the 'smaller' series $\sum_{k=1}^{\infty} a_k$ diverges, then, the 'bigger' series $\sum_{k=1}^{\infty} b_k$ also diverges.

Notes

Q: Use the comparison test to determine whether the following series converges or diverges

(i) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-\frac{1}{2}}$ (ii) $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$.

(i) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-\frac{1}{2}} = \sum_{k=1}^{\infty} a_k$.

let: $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} b_k$.

$$\sum_{k=1}^{\infty} b_k < \sum_{k=1}^{\infty} a_k \quad \left(\frac{1}{\sqrt{k}} < \frac{1}{\sqrt{k}-\frac{1}{2}}\right)$$

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent, so, it is enough to te

$b_k < a_k$ for comparison test.

$\Rightarrow$ smaller series is divergent.

By comparison test,

$\therefore \sum_{k=1}^{\infty} a_k$ (bigger series) is also divergent.

(ii) $\sum_{k=1}^{\infty} \frac{1}{2k^2+k} = \sum_{k=1}^{\infty} a_k$

Let $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{2k^2}$

$$2k^2 \le 2k^2+k$$

$$\frac{1}{2k^2} \ge \frac{1}{2k^2+k}$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{2k^2} \ge \sum_{k=1}^{\infty} \frac{1}{2k^2+k}.$$

$$\therefore \sum_{k=1}^{\infty} b_k \ge \sum_{k=1}^{\infty} a_k$$

we know that, $\sum_{k=1}^{\infty} \frac{1}{2k^2}$ is convergent, (p-series with $p>1$)

i.e., bigger series is convergent.

$\therefore$ smaller series is also convergent

i.e., $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{2k^2+k}$ is convergent.

Q: Test the convergence of the following:

(i) $\sum_{k=1}^{\infty} \frac{1}{3^k+1}$ (ii) $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ (iii) $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}\ln k}$.

(i) $\sum_{k=1}^{\infty} \frac{1}{3^k+1} = \sum_{k=1}^{\infty} a_k.$

Let $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{3^k}$

$$3^k + 1 > 3^k$$

$$\frac{1}{3^k+1} < \frac{1}{3^k}$$

$$\sum_{k=1}^{\infty} \frac{1}{3^k+1} < \sum_{k=1}^{\infty} \frac{1}{3^k}$$

We know that, $\sum_{k=1}^{\infty} \frac{1}{3^k}$ is a convergent series.

(Geometric series with $|r| < 1$)

By comparison test,

The bigger series $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{3^k}$ converges.

$\therefore$ the smaller series $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{1}{3^k+1}$ also converges

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(ii) Let $\sum_{k=2}^{\infty} \frac{1}{\ln k} = \sum_{k=2}^{\infty} a_k$

$$\sum_{k=2}^{\infty} \frac{1}{k} = \sum_{k=2}^{\infty} b_k$$

$$\ln k < k$$

$$\frac{1}{\ln k} > \frac{1}{k}$$

$$\sum_{k=2}^{\infty} \frac{1}{\ln k} > \sum_{k=2}^{\infty} \frac{1}{k}$$

$\sum_{k=2}^{\infty} \frac{1}{k}$ is a harmonic series.

$\therefore$ it is divergent.

By comparison test,

The smaller series, $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges.

$\therefore$ the bigger series, $\sum_{k=2}^{\infty} \frac{1}{\ln k}$ also diverges.

(iii) Let $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}\ln k} = \sum_{k=2}^{\infty} a_k$

$$\sum_{k=2}^{\infty} \frac{1}{k} = \sum_{k=2}^{\infty} b_k$$

$$\sqrt{k}\ln k < k$$

$$\frac{1}{\sqrt{k}\ln k} > \frac{1}{k}$$

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}\ln k} > \sum_{k=2}^{\infty} \frac{1}{k}$$

$\sum_{k=2}^{\infty} \frac{1}{k}$ is a harmonic series

$\therefore$ it is divergent.

By comparison test,

The smaller series, $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges. So, the bigger series, $\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}\ln k}$ also diverges.

4/10/2019

Limit Comparison Test

Let Σa_k and Σb_k be series with positive terms and suppose that $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$.

If ρ is finite, and $\rho > 0$, then the series both converge or both diverge.

Q: Test the convergence of the following:

(i) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$

Let $a_k = \frac{1}{\sqrt{k}+1}$

Set $b_k = \frac{1}{\sqrt{k}}$

Consider $\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$

$$= \lim_{k\to\infty} \frac{\frac{1}{\sqrt{k}+1}}{\frac{1}{\sqrt{k}}}$$

$$= \lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k}+1}$$

$$= \lim_{k\to\infty} \frac{1}{1+\frac{1}{\sqrt{k}}}$$

$$= \underline{\underline{1}}$$

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$\rho = 1 > 0$

~~Also~~ $\Rightarrow \rho$ is finite and positive.

~~$\therefore$ By limit comparison test,~~

since, Σb_k is divergent, then, by limit comparison test, Σa_k is also divergent.

(ii) $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$

Let $a_k = \frac{1}{2k^2+k}$

$b_k = \frac{1}{2k^2}$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{2k^2}{2k^2+k}$$

$\frac{k}{2k^2}$

$$= \lim_{k\to\infty} \frac{1}{1+\frac{1}{2k}}$$

$\rho = 1 > 0$

$\Rightarrow \rho$ is finite and positive.

Since, Σb_k is convergent, then, by limit comparison test, Σa_k is convergent.

(iii) $\sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2}$

Let $a_k = \frac{3k^3-2k^2+4}{k^7-k^3+2}$

$$b_k = \frac{3k^3}{k^7} = \frac{3}{k^4}$$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{3k^3-2k^2+4}{k^7-k^3+2} \times \frac{k^4}{3}$$

$$= \frac{1}{3} \lim_{k\to\infty} \frac{k^3\left(3-\frac{2}{k}+\frac{4}{k^3}\right)}{k^7\left(1-\frac{1}{k^4}+\frac{2}{k^7}\right)} \times k^4$$

$$= \frac{1}{3} \times 3 = \underline{\underline{1}}$$

$\rho = 1 > 0$

$\Rightarrow \rho$ is finite and positive.

Since, $\sum b_k$ is convergent, then, by limit comparison test, $\sum a_k$ is convergent.

Q: Test the convergence of (i) $\sum_{k=1}^{\infty}(\sqrt{k^2+1}-k)$.

Let $a_k = \frac{\sqrt{k^2+1}-k}{\sqrt{k^2+1}+k} \times (\sqrt{k^2+1}+k)$

$$= \frac{k^2+1-k^2}{\sqrt{k^2+1}+k}$$

$$a_k = \frac{1}{\sqrt{k^2+1}+k}$$

$$b_k = \frac{1}{k}$$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}+k}$$

$$= \lim_{k\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{k^2}}}$$

$$= \frac{1}{2}$$

$$\rho = \frac{1}{2} > 0$$

$\Rightarrow \rho$ is finite and positive

since, Σb_k is divergent, Σa_k is also divergent.

(ii) $\sum_{k=1}^{\infty} \sqrt{k^4+k} - k^2$

Let $a_k = \sqrt{k^4+k} - k^2$

$$= \left(\sqrt{k^4+k} - k^2\right) \times \frac{\left(\sqrt{k^4+k} + k^2\right)}{\left(\sqrt{k^4+k} + k^2\right)}$$

$$= \frac{k^4+k-k^4}{\left(\sqrt{k^4+k}+k^2\right)}$$

$$a_k = \frac{k}{\sqrt{k^4+k}+k^2}$$

$$b_k = \frac{k}{2k^2} = \frac{1}{2k}$$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{2k^2}{\sqrt{k^4+k}+k^2}$$

$$= \lim_{k\to\infty} \frac{2}{\sqrt{1+\frac{1}{k^3}}+1} = \frac{2}{2} = 1$$

$\rho = 1 > 0$

$\Rightarrow \rho$ is finite and positive

Since, Σb_k is divergent, then, by limit comparison test, Σa_k is divergent.

(iii) $\sum_{k=1}^{\infty} \sqrt[3]{k^3+1} - k$

$(a^3-b^3)=(a-b)(a^2+ab+b^2)$

Let $a_k = \sqrt[3]{k^3+1} - k = (k^3+1)^{1/3} - k$

$$= \frac{k^3+1-k^3}{(k^3+1)^{2/3}+k(k^3+1)^{1/3}+k^2}$$

$$a_k = \frac{1}{(k^3+1)^{2/3}+k(k^3+1)^{1/3}+k^2}$$

$$b_k = \frac{1}{3k^2}$$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{3k^2}{(k^3+1)^{2/3}+k(k^3+1)^{1/3}+k^2}$$

$$= \lim_{k\to\infty} \frac{3}{\left(1+\frac{1}{k^3}\right)^{2/3}+\left(1+\frac{1}{k^3}\right)^{1/3}+1}$$

$$= \frac{3}{3} = \underline{\underline{1}}$$

$\rho = 1 > 0$

$\Rightarrow \rho$ is finite and positive

Since Σb_k is convergent, by limit comparison test, Σa_k is convergent.

(iv) $\sum_{k=1}^{\infty} \frac{1}{k} \sin \frac{1}{k}$

Let $a_k = \frac{1}{k} \sin \frac{1}{k}$

$b_k = \frac{1}{k^2}$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \left(\frac{1}{k} \sin \frac{1}{k}\right) \times k^2$$

$$= \lim_{k\to\infty} k \sin \frac{1}{k}$$

$$= \lim_{k\to\infty} \frac{\sin \frac{1}{k}}{\frac{1}{k}}$$

$$= 1$$

$\rho = 1 > 0$

$\Rightarrow \rho$ is finite and positive.

Since, Σb_k is convergent, Σa_k is convergent

(v) $\sum_{k=1}^{\infty} \sin \frac{1}{k}$

Let $a_k = \sin \frac{1}{k}$

$b_k = \frac{1}{k}$

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$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} k \sin\frac{1}{k}$$

$$= \underline{\underline{1}}$$

$$\rho = 1 > 0$$

Since, Σb_k is divergent, by limit comparison test, Σa_k is divergent.

Q: Test the convergence of (i) $\frac{1}{1\cdot3\cdot5} + \frac{2}{3\cdot5\cdot7} + \frac{3}{5\cdot7\cdot9} + \cdots$

$$\frac{1}{1\cdot3\cdot5} + \frac{2}{3\cdot5\cdot7} + \frac{3}{5\cdot7\cdot9} + \cdots + \frac{k}{(2k-1)(2k+1)(2k+3)}$$

$$\therefore \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{k}{(2k-1)(2k+1)(2k+3)}$$

Let $a_k = \frac{k}{(2k-1)(2k+1)(2k+3)}$

$$b_k = \frac{k}{2k\times 2k\times 2k} = \underline{\underline{\frac{1}{k^2}}}$$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{k}{(2k-1)(2k+1)(2k+3)} \times k^2$$

$$= \lim_{k\to\infty} \frac{1}{\left(2-\frac{1}{k}\right)\left(2+\frac{1}{k}\right)\left(2+\frac{3}{k}\right)}$$

$$= \frac{1}{2\times2\times2} = \frac{1}{8}$$

$$\rho = \frac{1}{8} > 0$$

$\Rightarrow \rho$ is finite and positive

Since, Σb_k is convergent, by limit compar test, Σa_k is convergent.

(ii) $\frac{1}{4\cdot6} + \frac{\sqrt{3}}{6\cdot8} + \frac{\sqrt{5}}{8\cdot10} + ---$

$$a_k = \frac{\sqrt{2k-1}}{(2k+2)(2k+4)}$$

$$b_k = \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}$$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{k^{3/2}\sqrt{2k-1}}{(2k+2)(2k+4)}$$

$$= \lim_{k\to\infty} \frac{\sqrt{2-\frac{1}{k}}}{\left(2+\frac{2}{k}\right)\left(2+\frac{4}{k}\right)}$$

$$\rho = \frac{\sqrt{2}}{4} > 0$$

$\Rightarrow \rho$ is finite and positive.

since, Σb_k is convergent, then by limit comparison test, Σa_k is convergent.

Ratio Test

Let Σu_k be a series with positive terms and suppose that $\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$.

(a) If $\rho < 1$, the series converges.

(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.

(c) If $\rho = 1$, the series may converge or diverge, so that, another test must be done.

Root Test

Let Σu_k be a series with positive terms and suppose that $\rho = \lim_{k\to\infty} \sqrt[k]{u_k} = \lim_{k\to\infty} (u_k)^{1/k}$.

(a) If $\rho < 1$, the series converges.

(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.

(c) If $\rho = 1$, the series may converge or diverge so that another test must be tried.

17/10/2019

Q: Test the convergence of the following:

(i) $\sum_{k=1}^{\infty} \frac{1}{k!}$

Let $u_k = \frac{1}{k!}$

$u_{k+1} = \frac{1}{(k+1)!}$

$$\frac{u_{k+1}}{u_k} = \frac{k!}{(k+1)!} = \frac{k!}{k!(k+1)} = \frac{1}{k+1}$$

Consider, $\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1$

Here, the given series has positive terms and then try to apply ratio test.

$\therefore$ By ratio test,

the given series is convergent.

(ii) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

Here, the given series has positive terms and then try to apply ratio test.

Let $u_k = \frac{k}{2^k}$

$u_{k+1} = \frac{k+1}{2^{k+1}}$

$$\frac{u_{k+1}}{u_k} = \frac{k+1}{2^{k+1}} \times \frac{2^k}{k} = \frac{k+1}{2k}$$

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \frac{1}{2}\left(\lim_{k\to\infty} \frac{k+1}{k}\right)$$

$$= \frac{1}{2} \lim_{k\to\infty} 1+\frac{1}{k}$$

$$\rho = \frac{1}{2} < 1$$

∴ By ratio test,
the given series is convergent.

(iii) $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$

Let $u_k = \frac{(2k)!}{4^k}$

$$u_{k+1} = \frac{[2(k+1)]!}{4^{k+1}} = \frac{(2k+2)!}{4^{k+1}}$$

$$\frac{u_{k+1}}{u_k} = \frac{(2k+2)!}{4^{k+1}} \times \frac{4^k}{(2k)!}$$

$$= \frac{(2k+2)(2k+1)}{4} = \frac{(k+1)(2k+1)}{2}$$

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$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k}$$

$$= \frac{1}{2} \lim_{k\to\infty} (k+1)(2k+1)$$

$$\rho = \infty$$

$\therefore$ By ratio test, the series is divergent.

(iv) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

Let $u_k = \frac{k^k}{k!}$

$$u_{k+1} = \frac{(k+1)^{(k+1)}}{(k+1)!}$$

$$\frac{u_{k+1}}{u_k} = \frac{(k+1)^{(k+1)}}{(k+1)!} \times \frac{k!}{k^k}$$

$$= \frac{(k+1)^{(k+1)}}{k^k\,(k+1)}$$

$$= \frac{(k+1)^k}{k^k}$$

$$= \left(\frac{k+1}{k}\right)^k$$

$$= \left(1+\frac{1}{k}\right)^k$$

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$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e$$

$$\rho = \lim_{k\to\infty}\frac{u_{k+1}}{u_k} = e > 1$$

$\therefore$ By ratio test, the series is divergent.

(v) $\sum_{k=1}^{\infty}\frac{1}{2k-1}$

Let $u_k = \frac{1}{2k-1}$

$$u_{k+1} = \frac{1}{2(k+1)-1} = \frac{1}{2k+2-1} = \frac{1}{2k+1}$$

$$\frac{u_{k+1}}{u_k} = \frac{2k-1}{2k+1}$$

$$\rho = \lim_{k\to\infty}\frac{u_{k+1}}{u_k} = \lim_{k\to\infty}\frac{\left(1-\frac{1}{2k}\right)}{\left(1+\frac{1}{2k}\right)} = 1$$

$\therefore$ By ratio test, the series may converge or diverge. Here, the test fails.

Using limit comparison test;

$$a_k = \frac{1}{2k-1}$$

$$b_k = \frac{1}{k}$$

$$\rho = \lim_{k\to\infty}\frac{a_k}{b_k}$$

$$= \lim_{k \to \infty} \frac{k}{2k-1}$$

$$= \lim_{k \to \infty} \frac{1}{2-\frac{1}{k}} = \frac{1}{2} > 0$$

$\Rightarrow \rho$ is finite and positive

Since, Σb_k is divergent, by limit comparison test, Σa_k is divergent.

(vi) $\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$

$$u_k = \left(\frac{4k-5}{2k+1}\right)^k$$

$$\rho = \lim_{k \to \infty} \sqrt[k]{u_k}$$

$$= \lim_{k \to \infty} (u_k)^{1/k}$$

$$= \lim_{k \to \infty} \left[\left(\frac{4k-5}{2k+1}\right)^k\right]^{1/k}$$

$$= \lim_{k \to \infty} \frac{4k-5}{2k+1}$$

$$= \lim_{k \to \infty} \frac{4-\frac{5}{k}}{2+\frac{1}{k}}$$

$$= 2 > 1$$

$\therefore$ By root test,

the given series is divergent.

(vii) $\sum_{k=1}^{\infty} \frac{1}{[\ln(k+1)]^k}$

$$u_k = \left[\frac{1}{\ln(k+1)}\right]^k$$

$$\rho = \lim_{k\to\infty} (u_k)^{1/k}$$

$$= \lim_{k\to\infty} \left[\left(\frac{1}{\ln(k+1)}\right)^k\right]^{1/k}$$

$$= \lim_{k\to\infty} \frac{1}{\ln(k+1)}$$

$$= \frac{1}{\infty} = 0 < 1$$

$\therefore$ By root test, the series is convergent.

(viii) $\sum_{k=1}^{\infty} \left(\frac{1}{2k+3}\right)^{17}$

$$a_k = \left(\frac{1}{2k+3}\right)^{17}$$

$$u_{k+1} = \left(\cancel{\frac{1}{2k+2+3}}\right)^{17}$$

$$b_k = \frac{1}{k^{17}}$$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{1}{(2k+3)^{17}} \times k^{17}$$

$$= \lim_{k\to\infty} \left(\frac{k}{2k+3}\right)^{17}$$

$$= \lim_{k\to\infty} \left(\frac{1}{2+\frac{3}{k}}\right)^{17}$$

$$\rho = \frac{1}{2^{17}} > 0$$

$\Rightarrow \rho$ is finite and positive.

Since Σb_k is convergent, by limit comparison test, Σa_k is convergent.

(ix) $\sum_{k=1}^{\infty} k \times \frac{1}{3^k}$

$$u_k = \frac{k}{3^k}$$

$$u_{k+1} = \frac{k+1}{3^{k+1}}$$

$$\frac{u_{k+1}}{u_k} = \frac{k+1}{3^{k+1}} \times \frac{3^k}{k}$$

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$$= \frac{k+1}{3k} = \frac{1}{3} + \frac{1}{3k}$$

$$\rho = \lim_{k\to\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to\infty} \left(\frac{1}{3} + \frac{1}{3k}\right) = \frac{1}{3} < 1$$

$\therefore$ By ratio test,

the given series is convergent.

(x) $\sum_{k=1}^{\infty} \frac{(k!)10^k}{3^k}$

$$u_k = k! \left(\frac{10}{3}\right)^k$$

$$u_{k+1} = (k+1)! \left(\frac{10}{3}\right)^{k+1}$$

$$\frac{u_{k+1}}{u_k} = \frac{(k+1)! \left(\frac{10}{3}\right)^{k+1}}{k! \left(\frac{10}{3}\right)^k}$$

$$= \frac{k!(k+1) \left(\frac{10}{3}\right)^k \cdot \frac{10}{3}}{k! \left(\frac{10}{3}\right)^k}$$

$$= (k+1)\frac{10}{3}$$

$$\rho = \lim_{k\to\infty} \frac{10k}{3} + \frac{10}{3} = \infty$$

$\rho = \infty$, by ratio test,

$\hookrightarrow$ the series diverges.

H.W (xi) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

(xii) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$.

(xiii) $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k^2}$

Let $a_k = \frac{\tan^{-1} k}{k^2}$

Since, $\tan^{-1} k < \frac{\pi}{2}$

we have, $\frac{\tan^{-1} k}{k^2} < \frac{\pi}{2k^2}$

Let $b_k = \frac{\pi}{2k^2}$

$\therefore \sum b_k = \frac{\pi}{2} \sum \frac{1}{k^2}$

which is a p-series which is convergent

Also, $\sum a_k < \sum b_k$

$\therefore$ By comparison test, the smaller series

$\sum a_k$ is convergent.

(xiv) $\sum_{k=1}^{\infty} \cot^{-1} k^2$

(xv) $\sum_{k=1}^{\infty} \frac{5^k + k^2}{k! + 3}$

We have $k^2 < 5^k \ \forall k$

$$k! + 3 > k!$$

$$\frac{1}{k!+3} < \frac{1}{k!}$$

$$\therefore \frac{5^k + k^2}{k!+3} < \frac{2.5^k}{k!}$$

$$\text{Let } b_k = \frac{2.5^k}{k!}$$

Now, we can test the convergence or divergence of Σb_k by ratio test.

$$b_{k+1} = \frac{2.5^{k+1}}{(k+1)!}$$

$$\rho = \lim_{k\to\infty} \frac{b_{k+1}}{b_k}$$

$$= \lim_{k\to\infty} \frac{2.5^{k+1}}{(k+1)!} \times \frac{k!}{2.5^k}$$

$$= \lim_{k\to\infty} \frac{5}{k+1}$$

$$\rho = 0 < 1$$

~~$\Rightarrow \rho$ is finite and positive~~

By root test, the series is convergent.

The larger series Σb_k is convergent.

$\therefore$ By comparison test, the smaller series Σa_k is convergent.

(xi) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

$$u_k = \frac{k!}{k^k}$$

$$u_{k+1} = \frac{(k+1)!}{(k+1)^{k+1}}$$

$$\frac{u_{k+1}}{u_k} = \frac{(k+1)k!}{(k+1)^k \cdot (k+1)} \times \frac{k^k}{k!}$$

$$= \left(\frac{k}{k+1}\right)^k = \left(\frac{1}{1+\frac{1}{k}}\right)^k$$

$$\rho = \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^k} = \frac{1}{e} < 1$$

By ratio test, the series ~~may~~ converges.

~~diverge.~~

~~Hence, the test fails.~~

(xii) $\sum_{k=1}^{\infty} \left(\frac{k}{k+1}\right)^{k^2}$

$$u_k = \left(\frac{k}{k+1}\right)^{k^2}$$

$$\rho = \lim_{k\to\infty} (u_k)^{1/k}$$

$$= \lim_{k\to\infty} \left[\left(\frac{k}{k+1}\right)^{k^2}\right]^{1/k}$$

$$= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^{k}$$

$$= \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^k}$$

$$\rho = \frac{1}{e} < 1$$

∴ By root test,

the series is convergent.

(xiv) $\sum_{k=1}^{\infty} \cot^{-1} k^2$

$$a_k = \cot^{-1} k^2 = \tan^{-1} \frac{1}{k^2}$$

24/10/201

Alternating series

Series whose terms alternate between positive and negative, are called alternating series.

e.g., $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

In general, an alternating series can have the forms

(1) $\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$

OR

(2) $\sum_{k=1}^{\infty} (-1)^{k} a_k = -a_1 + a_2 - a_3 + \cdots$

where, a_k is positive terms.

Alternating Series Test

Leibnitz Test

An alternating series of either form (1) or form (2) converges if the following two conditions are satisfied:

(a) $a_1 \geq a_2 \geq a_3 \geq \cdots a_k \geq \cdots$

(b) $\lim_{k \to \infty} a_k = 0$

Q: Test the convergence of the following series:

1) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$

$$a_k = \frac{1}{k}$$

$$a_{k+1} = \frac{1}{k+1}$$

$$\frac{a_{k+1}}{a_k} = \frac{k}{k+1} < 1$$

$$a_{k+1} < a_k$$

$$a_k > a_{k+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$$

$\therefore$ By Leibnitz test,

the series converges.

2) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$

$$a_k = \frac{k+3}{k(k+1)}$$

$$a_{k+1} = \frac{k+4}{(k+1)(k+2)}$$

$$\frac{a_{k+1}}{a_k} = \frac{(k+4)}{(k+1)(k+2)} \times \frac{k(k+1)}{(k+3)}$$

$$\frac{a_{k+1}}{a_k} = \frac{k(k+4)}{(k+2)(k+3)} = \frac{k^2+4k}{k^2+5k+6}.$$

$$\frac{a_{k+1}}{a_k} = \frac{k^2+4k}{k^2+4k+k+6} < 1$$

$$a_{k+1} < a_k$$

$$a_k > a_{k+1}$$

$\therefore$ a_k is showing decreasing nature

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k+3}{k(k+1)}$$

$$= \lim_{k\to\infty} \frac{1+\frac{3}{k}}{k+1}$$

$$= \underline{\underline{0}}$$

The conditions are satisfied.

Hence, by Leibnitz theorem, the series is convergent.

H.W

Q: Test the convergence or divergence of following:

(1) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{5k}{3^k}$. (2) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{4k+1}$

(3) $\sum_{k=3}^{\infty} (-1)^{k} \frac{\ln k}{k}$ (4) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{\sqrt{k}+1}$ $\frac{\ln k}{k}$

$\frac{\ln k}{k}$ is a $\downarrow$sing sequence.

(5) $\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k}$ (6) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1}$

(7) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$ (8) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$

(9) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$

Absolute convergence:

A series $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$ is said to converge absolutely if the series of absolute values

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$$

converges and is said to diverge absolutely if the series of absolute values diverges.

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Q: Determine whether the following series converg absolutely:

1) $1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - - - -$

$$\sum u_k = 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - - - -$$

$$\sum_{k=1}^{\infty} |u_k| = 1 + \frac{1}{2} + \frac{1}{2^2} + - - -$$

This is a geometric series with

$$|r| = \frac{1}{2} < 1$$

$\therefore$ The series $\sum_{k=1}^{\infty} |u_k|$ is convergent.

$\therefore$ The given series is absolutely converge

2) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - - -$

$$\sum u_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - - - -$$

$$\sum_{k=1}^{\infty} |u_k| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + - - -$$

This is a harmonic series which is diver

$\therefore$ The series of absolute values is divergent

Hence, the given series is absolutely divergent.

~~Also~~ (Here the series is convergent and absolutely divergent).

Theorem:

If the series $\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \cdots + |u_k| + \cdots$ converges, then the series, $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \cdots + u_k + \cdots$ is also convergent.

e.g., consider $\sum u_k = 1 - \frac{1}{2} - \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} - \cdots$ is convergent, since, the series of absolute values $\sum |u_k| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ which is a geometric series that is convergent.

Then, by the above theorem, the given series is convergent.

Q: ~~Show that~~ Determine whether the following series converge:

1) $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$

$$\sum u_k = \sum \frac{\cos k}{k^2}$$

$$u_k = \frac{\cos k}{k^2}$$

$$\sum |u_k| = \sum_{k=1}^{\infty} \frac{|\cos k|}{|k^2|}$$

$|\cos k| \leq$

$$\frac{|\cos k|}{k^2} \leq \frac{1}{k^2}$$

Here, the bigger series $\sum \frac{1}{k^2}$ is convergent.

$\therefore$ the smaller series, $\sum \frac{|\cos k|}{k^2}$ is convergent

$\Rightarrow$ The series of absolute values converges.

Thus, the given series converges absolutely and hence converges.

Note:

Absolutely convergent series is convergent.

Conditional convergence.

A series that converges but diverges absolutely is said to converge conditionally (or to be conditionally convergent).

e.g., $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

H.W

1) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{5k}{3^k}$

$$a_k = \frac{5k}{3^k}$$

$$a_{k+1} = \frac{5(k+1)}{3^{k+1}} = \frac{5k+5}{3^k \cdot 3}$$

$$\frac{a_{k+1}}{a_k} = \frac{5k+5}{3^k \cdot 3} \times \frac{3^k}{5k}$$

$$\frac{a_{k+1}}{a_k} = \frac{5k+5}{15k} = \frac{k+1}{3k} < 1$$

$$a_{k+1} < a_k$$

$$a_k > a_{k+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{5k}{3^k}$$

$$= \lim_{k\to\infty} \frac{5}{3^k \log 3}$$

$$= \underline{\underline{0}}$$

Both the conditions of leibnitz test are satisfied.

∴ The series converges.

ii) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{4k+11}$

$$a_k = \frac{k+1}{4k+11}$$

$$a_{k+1} = \frac{k+2}{4k+15}$$

$$\frac{a_{k+1}}{a_k} = \frac{(k+2)}{(4k+15)} \cdot \frac{(4k+11)}{(k+1)}$$

$$\frac{a_{k+1}}{a_k} = \frac{4k^2+11k+8k+22}{4k^2+4k+15k+15}$$

$$\frac{a_{k+1}}{a_k} = \frac{4k^2+19k+22}{4k^2+19k+15} = \frac{4k^2+19k+15+7}{4k^2+19k+15}$$

$$a_{k+1} > a_k$$

$$a_k < a_{k+1}$$

$\therefore$ This is not a decreasing series.

Condition for leibnitz test is not satisfied.

$\therefore$ It is a diverging series.

3) $\sum_{k=1}^{\infty} (-1)^k \frac{\ln k}{k}$

$a_k = \frac{\ln k}{k}$

$\frac{\ln k}{k}$ is a decreasing sequence.

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\ln k}{k}$$

$$= \lim_{k\to\infty} \frac{\frac{1}{k}}{1}$$

$$= \underline{\underline{0}}$$

Both the conditions of leibnitz test are satisfied.

Hence, the given series is convergent.

4) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+1}{\sqrt{k}+1}$

$$\frac{(k+2)(\sqrt{k+1}-1)(k-1)}{k(k+1)(\sqrt{k}-1)}$$

$$a_k = \frac{k+1}{\sqrt{k}+1} = \frac{(k+1)(\sqrt{k}-1)}{k-1}$$

$$a_{k+1} = \frac{k+2}{\sqrt{k+1}+1} = \frac{(k+2)(\sqrt{k+1}-1)}{k}$$

$$\frac{a_{k+1}}{a_k} = \frac{k+2}{\sqrt{k+1}+1} \times \frac{\sqrt{k}+1}{k+1}$$

$a_{k+1} > a_k$

$a_k < a_{k+1}$

$\therefore$ a_k is not decreasing in nature.

Condition for leibnitz test is not satisfied.

Hence, the given series is divergent.

5) $\sum_{k=1}^{\infty} (-1)^{k+1} e^{-k}$

$a_k = e^{-k}$

$a_{k+1} = e^{-(k+1)}$

$$\frac{a_{k+1}}{a_k} = \frac{e^{-(k+1)}}{e^{-k}} = \frac{e^{-k} \cdot e^{-1}}{e^{-k}} = \frac{1}{e} < 1$$

$a_{k+1} < a_k$

$a_k > a_{k+1}$

$\therefore$ a_k is a decreasing sequence.

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} e^{-k}$$

$$= 0$$

Both the conditions of leibnitz theorem are satisfied.

Hence, the given series is convergent.

6) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1}$

$$a_k = \frac{1}{k+1}$$

$$a_{k+1} = \frac{1}{k+2}$$

$$\frac{a_{k+1}}{a_k} = \frac{k+1}{k+2} = \frac{k+1}{k+1+1} < 1$$

$$a_{k+1} < a_k$$

$$a_k > a_{k+1}$$

$\therefore a_k$ is decreasing in nature.

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k+1} = 0$$

Both the conditions of leibnitz theorem are satisfied.

Hence, the given series is convergent.

7) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$

$$a_k = \frac{k+3}{k(k+1)}$$

$$a_{k+1} = \frac{k+4}{\cancel{k(k}(k+1)(k+2)}$$

$$\frac{a_{k+1}}{a_k} = \frac{(k+4)k\cancel{(k+1)}}{\cancel{(k+1)}(k+2)(k+3)}$$

$$= \frac{k^2+4k}{k^2+5k+6} = \frac{k^2+4k}{k^2+4k+k+6} < 1$$

$$a_{k+1} < a_k$$

$$a_k > a_{k+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k+3}{k(k+1)} = 0$$

∴ By leibnitz test, the given series is convergent.

8) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$

$$a_k = \frac{1}{k}$$

$$a_{k+1} = \frac{1}{k+1}$$

$$\frac{a_{k+1}}{a_k} = \frac{k}{k+1} < 1$$

$$a_k > a_{k+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$$

$\therefore$ By leibnitz test, the given series converges.

9) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2}$

$$a_k = \frac{1}{k^2}$$

$$a_{k+1} = \frac{1}{(k+1)^2}$$

$$\frac{a_{k+1}}{a_k} = \frac{k^2}{(k+1)^2} = \frac{k^2}{k^2+2k+1} < 1$$

$$a_{k+1} < a_k$$

$$a_k > a_{k+1}$$

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$\therefore a_k$ is a decreasing in nature.

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k^2}$$

$$= 0$$

Both the conditions of Leibnitz theorem are satisfied.

Hence, the given series converges.

28/10/2019

Q: Determine whether the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)}$ converges absolutely or converges conditionally

We test the series for absolute convergence first:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+3}{k(k+1)} = \sum u_k$$

$$|u_k| = \left|\frac{k+3}{k(k+1)}\right| = \frac{k+3}{k(k+1)}$$

$$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \frac{k+3}{k(k+1)} = \sum a_k$$

$$b_k = \frac{1}{k}$$

$$\sum b_k = \sum \frac{1}{k}$$

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$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{k+3}{k(k+1)} \times k$$

$\rho = 1 > 0$ and finite.

Σb_k is divergent series.

$\therefore$ By limit comparison test, Σa_k is also divergent.

$\therefore$ Given series is absolutely divergent.

Now, we test the conditional convergence of gn series

Let $u_k = \dfrac{k+3}{k(k+1)}$

$$u_{k+1} = \frac{k+4}{(k+1)(k+2)}$$

$$\frac{u_{k+1}}{u_k} = \frac{k+4}{(k+1)(k+2)} \times \frac{k(k+1)}{(k+3)}$$

$$= \frac{k^2+4k}{k^2+4k+k+6} < 1$$

$$\frac{u_{k+1}}{u_k} < 1$$

$$u_{k+1} < u_k$$

$\therefore$ The terms of series are decreasing in nature.

Also, $\lim_{k\to\infty} u_k = \lim_{k\to\infty} \frac{k+3}{k(k+1)} = 0$

Both the conditions of leibnitz test are satisfied by Σu_k.

Hence, the series is convergent.

Thus, we have shown that the given series is absolutely divergent and also it is convergent

Hence, the given series is conditionally convergent.

Ratio Test for Absolute Convergence.

Let Σu_k be a series with nonzero terms and suppose that

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|}$$

(a) If $\rho < 1$, then the series Σu_k converges absolutely and therefore converges.

(b) If $\rho > 1$ or if $\rho = +\infty$, then the series Σu_k diverges.

(c) If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.

Q: ~~Test~~ Determine the convergence or divergence of the following series:

1) $\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$ 2) $\sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{3^k}$

---

1) Let $u_k = \frac{2^k}{k!}(-1)^k$

$$|u_k| = \left| (-1)^k \frac{2^k}{k!} \right| = \frac{2^k}{k!}$$

$$|u_{k+1}| = \frac{2^{k+1}}{(k+1)!}$$

By ratio test,
$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|}$$

$$= \lim_{k\to\infty} \frac{2^{k+1}}{(k+1)!} \times \frac{k!}{2^k}$$

$$= \lim_{k\to\infty} \frac{2}{k+1}$$

$$\rho = 0 < 1$$

By ratio test for absolute convergence,
$\therefore$ The series $\sum u_k$ converges absolutely and therefore, the given series ~~and then~~ converges.

2) $\sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{3^k}$.

$$u_k = (-1)^k \frac{(2k-1)!}{3^k}$$

$$|u_k| = \frac{(2k-1)!}{3^k}$$

$$|u_{k+1}| = \frac{[2(k+1)-1]!}{3^{k+1}}$$

$$= \frac{(2k+1)!}{3^{k+1}}$$

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|}$$

$$= \lim_{k\to\infty} \frac{(2k+1)!}{3^k \cdot 3} \times \frac{3^k}{(2k-1)!}$$

$$= \lim_{k\to\infty} \frac{2k(2k+1)}{3}$$

$$\cancel{= \lim} = \frac{2}{3} \lim_{k\to\infty} 2k^2+k.$$

$$= +\infty$$

$\rho = +\infty$

$\therefore$ By ratio test for absolute convergence,

the series $\sum u_k$ diverges.

Q: ~~Det~~ Classify each series as absolutely convergent, conditionally convergent or divergent:

(1) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4/3}}$ (2) $\sum_{k=3}^{\infty} (-1)^k \frac{\ln k}{k}$ ~~(3) $\sum \sin kx$~~

(3) $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots + (-1)^{k+1} \frac{\sin kx}{k^3} + \cdots \infty.$

(1) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{4/3}}$

$$u_k = \frac{(-1)^{k+1}}{k^{4/3}}$$

$$|u_k| = \frac{1}{k^{4/3}}$$

$$|u_{k+1}| = \frac{1}{(k+1)^{4/3}}$$

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|}$$

$$= \lim_{k\to\infty} \frac{k^{4/3}}{(k+1)^{4/3}}$$

$$= \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^{4/3}}$$

$$= 1$$

$\Sigma|u_k| = \frac{1}{k^{4/3}}$ is a p-series with $p>1$

$\therefore$ The series ~~is~~ converges absolutely.

(2) Let $u_k = \frac{(-1)^k \ln k}{k}$

$$|u_k| = \frac{\ln k}{k}$$

$$\therefore \sum_{k=3}^{\infty} |u_k| = \sum_{k=3}^{\infty} \frac{\ln k}{k}$$

$\boxed{\frac{\ln k}{k} > \frac{1}{k}}$ for $k \geq 3$

$$\therefore \Sigma \frac{\ln k}{k} > \Sigma \frac{1}{k}$$

Now, $\Sigma \frac{1}{k}$ is divergent.

$\therefore$ By comparison test, $\Sigma |u_k|$ is divergent.

$\therefore \Sigma u_k$ absolutely diverges.

Now, we can check the conditional convergence of Σu_k.

$\frac{\ln k}{k}$ is decreasing.

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{\ln k}{k} = \lim_{k\to\infty} \frac{1}{k} = 0$$

$\therefore$ By leibnitz test,

Σu_k is convergent.

Hence, the given series is conditionally convergent.

30/10/2019

(3) $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \cdots + (-1)^{k+1} \frac{\sin kx}{k^3} + \cdots \infty$

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k^3}$$

$$u_k = (-1)^{k+1} \frac{\sin kx}{k^3}$$

$$|u_k| = \frac{|\sin kx|}{|k^3|}$$

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$$|\sin kx| < 1$$

$$\frac{|\sin kx|}{k^3} < \frac{1}{k^3}$$

$$\sum \frac{|\sin kx|}{k^3} < \sum \frac{1}{k^3}$$

$\sum \frac{1}{k^3}$ is a p-series with $p = 3 > 1$.

Hence, it is a convergent series.

$\therefore$ By comparison test, the bigger series $\sum \frac{1}{k^3}$ converges, hence, the smaller series $\sum |u_k|$ also converges.

$\therefore \sum u_k$ is absolutely convergent.

Q: Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{(2n-1)}}{n^2+1}$ is absolutely convergent.

Let $\sum u_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^{(2n-1)}}{n^2+1}$

$$u_n = (-1)^{n+1} \frac{3^{(2n-1)}}{n^2+1}$$

$$|u_n| = \frac{3^{(2n-1)}}{n^2+1}$$

$$\Sigma|u_n| = \Sigma \frac{3^{(2n-1)}}{n^2+1} = \Sigma V_n$$

$$V_n = \frac{3^{(2n-1)}}{n^2+1}$$

$$V_{n+1} = \frac{3^{(2n+1)}}{(n+1)^2+1}$$

$$\rho = \lim_{n\to\infty} \frac{V_{n+1}}{V_n} = \lim_{n\to\infty} \frac{3^{2n}\cdot 3}{(n+1)^2+1} \times \frac{(n^2+1)3}{3^{2n}}$$

$$= \lim_{n\to\infty} \frac{9(n^2+1)}{n^2+2n+2}$$

$$= 9 \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}$$

$$= 9$$

$$\rho = 9 > 1$$

Hence, by ratio test, Σu_k is divergent.

31/10/2019

Q: Determine whether the series $\sum_{k=1}^{\infty}(-1)^{k+1}.\frac{1}{7k}$ is absolutely convergent, conditionally convergent or divergent.

Let $\sum u_k = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{7k}$

$$|u_k| = \frac{1}{7k}$$

$$|u_{k+1}| = \frac{1}{7k+7}$$

$$\lim_{k\to\infty}\frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty}\frac{7k}{7k+7} = 1$$

$\therefore$ The ratio test fails.

$$\sum_{k=1}^{\infty}|u_k| = \sum_{k=1}^{\infty}\frac{1}{7k} = \frac{1}{7}\left(\sum_{k=1}^{\infty}\frac{1}{k}\right)$$

$\because \sum_{k=1}^{\infty}\frac{1}{k}$ is divergent, $\sum_{k=1}^{\infty}|u_k|$ is divergent.

Hence, $\sum u_k$ is absolutely divergent.

$$a_k = \frac{1}{7k}$$

$$a_{k+1} = \frac{1}{7k+7}$$

$$\frac{a_{k+1}}{a_k} = \frac{7k}{7k+7} < 1$$

$$a_{k+1} < a_k$$

$\Rightarrow$ The series has decreasing nature.

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{7k} = 0$$

Both the conditions of Leibnitz test are satisfied.

Hence, the series is convergent.

$\therefore$ The given series is conditionally convergent.

H.W

Q: Classify each series as absolutely convergent, conditionally convergent or divergent:

(1) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$

Let $\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$

$$|u_k| = \frac{1}{k!}$$

$$|u_{k+1}| = \frac{1}{(k+1)!}$$

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$$\rho = \lim_{k \to \infty} \frac{|u_{k+1}|}{|u_k|}$$

$$= \lim_{k \to \infty} \frac{k!}{(k+1)!}$$

$$= \lim_{k \to \infty} \frac{1}{k+1}$$

$$\rho = 0 < 1$$

$\therefore$ By ratio test for absolute convergence, the given series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}$ converges absolutely

---

(2) $\sum_{k=1}^{\infty} \frac{\sin(2k+1)\frac{\pi}{2}}{k}$

Let $\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \frac{\sin(2k+1)\frac{\pi}{2}}{k}$

sin(2k

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \frac{\cos k\pi}{k}$$

$$\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

$$\sum_{k=1}^{\infty} |u_k| = \sum_{k=1}^{\infty} \left|\frac{(-1)^k}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$$

$u_k = \frac{1}{k}$, $u_{k+1} = \frac{1}{k+1}$

$$\frac{u_{k+1}}{u_k} = \frac{k}{k+1} < 1$$

$$u_{k+1} < u_k$$

$\therefore$ The series is decreasing in nature.

$$\lim_{k\to\infty} u_k = \lim_{k\to\infty} \frac{1}{k} = 0$$

Hence, conditions of Leibnitz test are satisfied.

$\therefore$ The series is convergent.

Also, $\Sigma|u_k| = \Sigma\frac{1}{k}$ is a divergent series

$\therefore \Sigma u_k$ is absolutely divergent.

Hence, the given series is conditionally convergent

(3) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+7}{k(k+4)}$

Let $\sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k+7}{k(k+4)}$

$$|u_k| = \frac{k+7}{k(k+4)}$$

$$|u_{k+1}| = \frac{k+8}{(k+1)(k+5)}$$

$$\frac{|u_{k+1}|}{|u_k|} = \frac{(k+8)\,k\,(k+4)}{(k+1)(k+5)(k+7)}$$

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|}$$

$$= \lim_{k\to\infty} \frac{k^3\left(1+\frac{8}{k}\right)\left(1+\frac{4}{k}\right)}{k^3\left(1+\frac{1}{k}\right)\left(1+\frac{5}{k}\right)\left(1+\frac{7}{k}\right)}$$

$$= \underline{\underline{1}}$$

$\therefore$ The test fails.

Let $\sum a_k = \sum \frac{k+7}{k(k+4)}$

$$\sum b_k = \sum \frac{k}{k^2} = \sum \frac{1}{k}.$$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{(k+7)k}{k(k+4)}$$

$$= \lim_{k\to\infty} \frac{1+\frac{7}{k}}{1+\frac{4}{k}} = 1 \cancel{<\infty} > 0$$

$\Rightarrow$ f is finite and positive.

since, Σb_k is divergent, by limit comparison test, Σa_k is ~~con~~divergent.

$\Rightarrow \Sigma |u_k|$ is ~~con~~divergent

$\therefore$ The given series is absolutely divergent.

$$a_k = \frac{k+7}{k(k+4)}$$

$$a_{k+1} = \frac{k+8}{(k+1)(k+5)}$$

$$\frac{a_{k+1}}{a_k} = \frac{(k^2+8k)(k+4)}{(k^2+6k+5)(k+7)}$$

$$= \frac{k^3+4k^2+8k^2+32k}{k^3+7k^2+6k^2+42k+5k+35}$$

$$\frac{a_{k+1}}{a_k} = \frac{k^3+12k^2+32k}{k^3+12k^2+32k+k^2+15k+35} < 1$$

$$a_{k+1} < a_k$$

$\therefore$ The series is decreasing in nature.

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k+7}{k(k+4)}$$

$$= \lim_{k\to\infty} \frac{1+\frac{7}{k}}{k+4}$$

$$= \underline{\underline{0}}$$

Both the conditions of Leibnitz test are satisfied.

Hence, the given series is convergent.

$\therefore$ The series, Σu_k is conditionally converge

(iv) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k^3+1}$.

Let $\Sigma u_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{k^2}{k^3+1}$

$$\Sigma |u_k| = \Sigma \frac{k^2}{k^3+1} = \Sigma a_k$$

$$\Sigma b_k = \Sigma \frac{k^2}{k^3} = \Sigma \frac{1}{k}$$

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{k^2}{k^3+1} \times k$$

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$$= \lim_{k\to\infty} \frac{1}{1+\frac{1}{k^3}} = 1 > 0$$

ℓ is finite and positive.

Since, Σb_k is divergent, by limit comparison test, Σa_k is also divergent.

$\Rightarrow \Sigma |u_k|$ is divergent

$\Rightarrow \Sigma u_k$ is absolutely divergent.

$$a_k = \frac{k^2}{k^3+1}$$

$$a_{k+1} = \frac{(k+1)^2}{(k+1)^3+1}$$

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(k+1)^3+1} \times \frac{(k^3+1)}{k^2}$$

$$= \frac{(k^2+2k+1)(k^3+1)}{[(k+1)^3+1]\,k^2}$$

$$= \frac{k^5+2k^4+k^3+k^2+2k+1}{k^5+2k^4+k^3+k^2+(k^4+2k^3)} < 1$$

$a_{k+1} < a_k$

$\therefore$ The series is decreasing in nature.

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^3+1}$$

$$= \lim_{k\to\infty} \frac{1}{k+\frac{1}{k^2}}$$

$$= \underline{0}$$

Both the conditions of Leibnitz test are satisfied.

Hence, the series is convergent.

$\therefore$ The given series is conditionally convergent

(v) $\sum_{k=1}^{\infty} \sin \frac{k\pi}{2}$

Let $\sum a_k = \sum_{k=1}^{\infty} \sin \frac{k\pi}{2}$

$\cancel{a_k = \sin \frac{k\pi}{2}}$

$$a_k = \sin \frac{k\pi}{2}$$

$$a_{k+1} = \sin (k+1)\frac{\pi}{2}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \sin \frac{k\pi}{2}$$

$$= \lim_{k\to\infty} \frac{\sin\frac{k\pi}{2}}{\frac{k\pi}{2}} \times \frac{k\pi}{2}$$

$$= \infty \neq 0$$

Hence, by k^{th} term test, the given seeies is divergent.

(vi) $\sum_{k=1}^{\infty} \frac{k\cos k\pi}{k^2+1}$

$$a_k = \frac{k\cos k\pi}{k^2+1}$$

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k\cos k\pi}{k^2+1}$$

$$= \lim_{k\to\infty} \frac{\frac{\cos k\pi}{k\pi} \times k^2\pi}{k^2+1}$$

$$= \pi \lim_{k\to\infty} \frac{1}{1+\frac{1}{k^2}}$$

$$= \pi \neq 0$$

$\therefore$ By k^{th} term test, the given series is divegent.

(vii) $\sum_{k=2}^{\infty}\left(\frac{-1}{\ln k}\right)^{k}$.

Let $\sum_{k=2}^{\infty} u_k = \sum_{k=2}^{\infty}\left(\frac{-1}{\ln k}\right)^{k} \Rightarrow \sum|u_k| = \sum\left(\frac{1}{\ln k}\right)^{k} = \sum a_k$

$$\rho = \lim_{k\to\infty} (a_k)^{1/k}$$

$$= \lim_{k\to\infty}\left(\frac{1}{\ln k}\right)^{k\times 1/k}$$

$$= \lim_{k\to\infty}\frac{1}{\ln k}$$

$$= \frac{1}{\infty} = 0 < 1$$

By root test, the series is convergent.

$$|u_k| = \left|\left(\frac{-1}{\ln k}\right)^{k}\right| = \left(\frac{1}{\ln k}\right)^{k}$$

$$|u_{k+1}| = \left|\left[\frac{-1}{\ln(k+1)}\right]^{k+1}\right| = \left[\frac{1}{\ln(k+1)}\right]^{k+1}$$

Hence, by root test,

$\sum a_k$ is convergent ($\because \rho < 1$)

$\Rightarrow \Sigma|u_k|$ is convergent

$\therefore$ The given series $\Sigma c_k = \sum_{k=2}^{\infty} \left(\frac{-1}{\ln k}\right)^k$

is absolutely convergent.

(viii) $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$.

Let $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \frac{\sin k}{k^2}$.

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{k}$.

$$\rho = \lim_{k\to\infty} \frac{a_k}{b_k}$$

$$= \lim_{k\to\infty} \frac{\sin k}{k^2} \times k$$

$$= \lim_{k\to\infty} \frac{\sin k}{k}$$

$$= 1 > 0$$

Since, Σb_k is divergent, by limit comparison test, Σa_k is also divergent.

31/10/2019

MODULE - V :

Taylor's Series :

If $f(x)$ has derivatives of all orders at x_0, then, we call the series $f(x) = \sum_{k=0}^{\infty} \frac{f^k(x_0)}{k!}(x-x_0)^k$

$$= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$+ \frac{f^n(x_0)}{n!}(x-x_0)^n + \cdots + \infty$$

The Taylor series for $f(x)$ about the point $x = x_0$.

Note:

In the special case, where $x_0 = 0$, the above series become $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots +$

$$\frac{f^n(0)}{n!}x^n + \cdots + \infty$$

In this case, we call this Maclaurin's series for $f(x)$.

Q: Write the Taylor series for $\frac{1}{x}$ about $x = 1$.

Take $f(x) = \frac{1}{x}$

$x_0 = 1$

In the Taylor series expansion,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f''''(x_0)}{4!}(x-x_0)^4 + \frac{f^{V}(x_0)}{5!}(x-x_0)^5 + \cdots \infty$$

$$\frac{1}{x} = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{IV}(1)}{4!}(x-x_0)^4 + \frac{f^{V}(1)}{5!}(x-1)^5 + \cdots \infty.$$

$$\frac{1}{x} = 1 + (-1)(x-1) + \frac{2}{2}(x-1)^2 + \left(\frac{-6}{6}\right)(x-1)^3 + \frac{24}{24}(x-1)^4 + \frac{(-120)}{120}(x-1)^5 + \cdots \infty$$

$$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5 + \cdots \infty.$$

H.W

Q: Find the Taylor series of

(i) $f(x) = e^x$ about $x_0 = -1$

(ii) $\ln x$ about $x_0 = 1$

(iii) $f(x) = \frac{1}{x+2}$ about $x_0 = 1$

(iv) $f(x) = e^{-x}$ about $x_0 = \ln 3$

(v) $f(x) = \cos x$ about $x_0 = \frac{\pi}{2}$.

(i) $f(x) = e^x$ about $x_0 = -1$.

In the Taylor series expansion,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$$+ \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{IV}(x_0)}{4!}(x-x_0)^4 + \frac{f^{V}(x_0)}{5!}(x-x_0)^5 + \cdots$$

$$e^x = f(-1) + \frac{f'(-1)}{1!}(x+1) + \frac{f''(-1)}{2!}(x+1)^2$$

$$+ \frac{f'''(-1)}{3!}(x+1)^3 + \frac{f^{IV}(x_0)}{4!}(x+1)^4 + \frac{f^{V}(x_0)}{5!}(x+1)^5 +$$

$$e^x = e^{-1} + \frac{e^{-1}}{1!}(x+1) + \frac{e^{-1}}{2!}(x+1)^2 + \frac{e^{-1}}{3!}(x+1)^3$$

$$+ \frac{e^{-1}}{4!}(x+1)^4 + \frac{e^{-1}}{5!}(x+1)^5 + \cdots + \infty$$

$$e^x = \frac{1}{e}\left[1 + \frac{x+1}{1!} + \frac{(x+1)^2}{2!} + \frac{(x+1)^3}{3!} + \frac{(x+1)^4}{4!} + \cdots\right.$$

(ii) $\ln x$ about $x_0 = 1$

Take $f(x) = \ln x$

$x_0 = 1$

In the Taylor series expansion,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$+ \frac{f^{IV}(x_0)}{4!}(x-x_0)^4 + \frac{f^{V}(x_0)}{5!}(x-x_0)^5 + \cdots \infty$$

$$\ln x = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$+ \frac{f^{IV}(1)}{4!}(x-1)^4 + \frac{f^{V}(1)}{5!}(x-1)^5 + \cdots \infty$$

$$\ln x = 0 + \frac{x-1}{1!} + \frac{(-1)}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$+ \frac{(-6)}{4!}(x-1)^4 + \frac{24}{5!}(x-1)^5 + \cdots \infty$$

$$\ln x = \frac{x-1}{1} + \frac{(-1)(x-1)^2}{2} + \frac{2(x-1)^3}{3\times 2} + \frac{(-6)(x-1)^4}{4\times 3\times 2}$$

$$+ \frac{24(x-1)^5}{5\times 4\times 3\times 2} + \cdots \infty$$

$$\ln x = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} + \cdots + \infty$$

(iii) $f(x) = \frac{1}{x+2}$ about $x_0 = 1$

In the Taylor series expansion,

$$f(x) = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f^{IV}(x_0)}{4!}(x-x_0)^4 + \frac{f^{V}(x_0)}{5!}(x-x_0)^5 + \dots$$

$$\frac{1}{x+2} = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{IV}(1)}{4!}(x-1)^4 + \frac{f^{V}(1)}{5!}(x-1)^5 + \dots \infty$$

$$\frac{1}{x+2} = \frac{1}{3} + \left(\frac{-1}{3^2}\right)(x-1) + \left(\frac{1}{3^3}\right)(x-1)^2 + \left(\frac{-1}{3^4}\right)(x-1)^3 + \left(\frac{1}{3^5}\right)(x-1)^4 + \left(\frac{-1}{3^6}\right)(x-1)^5 + \dots \infty$$

$$\frac{1}{x+2} = \frac{1}{3} - \frac{x-1}{3^2} + \frac{(x-1)^2}{3^3} - \frac{(x-1)^3}{3^4} + \frac{(x-1)^4}{3^5} - \frac{(x-1)^5}{3^6} + \dots + \infty$$

(iv) $f(x) = e^{-x}$ about $x_0 = \ln 3$.

In the Taylor series expansion,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$+ \frac{f^{IV}(x_0)}{4!}(x-x_0)^4 + \frac{f^{V}(x_0)}{5!}(x-x_0)^5 + \cdots + \infty$$

$$e^{-x} \cancel{f(x)} = f(\ln 3) + \frac{f'(\ln 3)}{1!}(x-\ln 3) + \frac{f''(\ln 3)}{2!}(x-\ln 3)^2$$

$$+ \frac{f'''(\ln 3)}{3!}(x-\ln 3)^3 + \frac{f^{IV}(\ln 3)}{4!}(x-\ln 3)^4 + \cdots + \infty$$

$$e^{-x} = e^{-\ln 3} + \frac{(-e^{-\ln 3})}{1!}(x-\ln 3) + \frac{e^{-\ln 3}}{2!}(x-\ln 3)^2$$

$$+ \frac{(-e^{-\ln 3})}{3!}(x-\ln 3)^3 + \frac{(e^{-\ln 3})}{4!}(x-\ln 3)^4 + \cdots + \infty$$

$$e^{-x} = e^{-\ln 3}\left[1 - \frac{x-\ln 3}{1!} + \frac{(x-\ln 3)^2}{2!} - \frac{(x-\ln 3)^3}{3!} + \frac{(x-\ln 3)^4}{4!} + \cdots + \infty\right]$$

$$e^{-x} = \frac{1}{3}\left[1 - \frac{(x-\ln 3)}{1!} + \frac{(x-\ln 3)^2}{2!} - \frac{(x-\ln 3)^3}{3!} + \frac{(x-\ln 3)^4}{4!} + \cdots \infty\right]$$

(v) $f(x) = \cos x$ about $x_0 = \frac{\pi}{2}$.

In the Taylor series expansion,

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3$$

$$+ \frac{f^{IV}(x_0)}{4!}(x-x_0)^4 + \frac{f^{V}(x_0)}{5!}(x-x_0)^5 + \cdots + \infty$$

$$\cos x = f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x-\frac{\pi}{2}\right) + \frac{f''(\pi/2)}{2!}\left(x-\frac{\pi}{2}\right)^2$$

$$+ \frac{f'''(\pi/2)}{3!}\left(x-\frac{\pi}{2}\right)^3 + \frac{f^{IV}(\pi/2)}{4!}\left(x-\frac{\pi}{2}\right)^4 + \frac{f^{V}(\pi/2)}{5!}\left(x-\frac{\pi}{2}\right)^5 + \cdots + \infty$$

$$\cos x = 0 + \left(\frac{-1}{1!}\right)\left(x-\frac{\pi}{2}\right) + \left(\frac{0}{2!}\right) + \left(\frac{1}{3!}\right)\left(x-\frac{\pi}{2}\right)^3$$

$$+ (0) + \left(\frac{-1}{5!}\right)\left(x-\frac{\pi}{2}\right)^5 + \cdots \infty$$

$$\cos x = -\frac{\left(x-\frac{\pi}{2}\right)}{1!} + \frac{\left(x-\frac{\pi}{2}\right)^3}{3!} - \frac{\left(x-\frac{\pi}{2}\right)^5}{5!} + \cdots \infty$$

1/11/2019

Power Series in x

If $c_0, c_1, c_2, \ldots$ are constants and x is a variable, then a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \ldots$$

is called power series in x with some coefficients $c_0, c_1, c_2, \ldots$

e.g., $\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Radius And Interval of convergence

If a numerical value is substituted for 'x' in a power series $\sum_{k=0}^{\infty} c_k x^k$. This leads to the problem of ~~exter~~ discovering the set of x values for which a given power series converge. This is called its convergence test.

Theorem:

For any power series, exactly one of the following is true:

(a) The series converges only for $x=0$.

(b) The series converges absolutely and hence converges for all values of x.

(c) The series converges absolutely and hence converges for all x in some finite open interval $(-R, R)$ and diverges if $x < -R$ or $x > R$.

At either of the values $x=R$ or $x=-R$, the series may converge absolutely, converge conditionally or diverge depending on the particular series.

Note:

1. The above theorem states that the convergence set for a power series in x is always an interval centered at $x=0$. For this reason, the convergence set of a power series in x is called the interval of convergence.

2. In the case where, the convergence set is a single value $x=0$, we say that the

series has radius of convergence zero. In the case where the convergence set is $(-\infty, \infty)$, we say that the power series has radius of convergence $+\infty$ and in the case where convergence set is $(-R, R)$, we say that the power series has radius of convergence R.

Radius of convergence $= 0$

Radius of convergence $= +\infty$

Radius of convergence $= R$

Finding the interval of convergence.

~~We can find~~

The usual procedure for finding the interval of convergence for a power series is to apply ratio test for absolute convergence.

Q: Find the interval of convergence and radius of convergence of the following series:

(i) $\sum_{k=0}^{\infty} x^k$ (ii) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ (iii) $\sum_{k=0}^{\infty} (k!) x^k$ (iv) $\sum_{k=0}^{\infty} \frac{(-1)^k x^k}{3^k (k+1)}$

Applying the ratio test for absolute convergence of the given series

(i) $\sum_{k=0}^{\infty} x^k$

$$|u_{k+1}| = |x^{k+1}|$$

$$|u_k| = |x^k|$$

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \frac{|x^{k+1}|}{|x^k|}$$

$$= \lim_{k\to\infty} \left| \frac{x^{k+1}}{x^k} \right|$$

$$= \lim_{k\to\infty} |x| = \underline{\underline{|x|}}$$

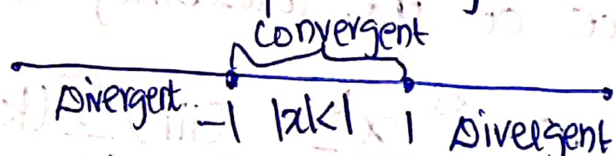
If $\rho < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1$

If $\rho > 1 \Rightarrow |x| > 1$

The test is inconclusive if $|x| = 1$.

i.e., $|x| = 1$ or -1

which we have to investigate convergences of these values separately.



At $x = -1 \Rightarrow \sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - - - - \Rightarrow$ Divergent

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At $x=1 \Rightarrow \sum_{k=0}^{\infty} 1^k = 1+1+1+\cdots \Rightarrow$ Divergent

The series converges at $(-1,1)$.

Thus the interval of convergence is $(-1,1)$ and

$\therefore$ radius of convergence $R=1$.

(ii) $\sum_{k=0}^{\infty} \frac{x^k}{k!}$

By ratio test for absolute convergence,

$$\rho = \lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)!} \times \frac{k!}{x^k} \right|$$

$$= \lim_{k\to\infty} \frac{x}{k+1} = \underline{\underline{0}}$$

$\therefore$ It is convergent. $-\infty$ —— ∞.

$\therefore$ The convergence interval is $(-\infty,\infty)$.

$\rho<1$

$\therefore$ The series is convergent by ratio test.

(iii) $\sum_{k=0}^{\infty} k!\, x^k$.

$$\rho = \lim_{k\to\infty} \left| \frac{(k+1)!\, x^{k+1}}{k!\, x^k} \right| = \lim_{k\to\infty} |(k+1)x|$$

$$= \underline{\underline{+\infty}}$$

$\therefore$ The series is divergent.

According to ratio test the series is divergent for all non-zero values of x.

The radius of convergence, $R=0$.

$\therefore$ It is convergent only at the point $x=0$

5/11/2019

Q: Find the radius of convergence (interval of convergence) of the following power series:

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{3^n}$.

$$u_n = \frac{(-1)^n (x-4)^n}{3^n}$$

$$u_{n+1} = \frac{(-1)^{n+1} (x-4)^{n+1}}{3^{n+1}}$$

$$\rho = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}} \times \frac{3^n}{(-1)^n (x-4)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x-4}{3} \right|$$

$$= \left| \frac{x-4}{3} \right|$$

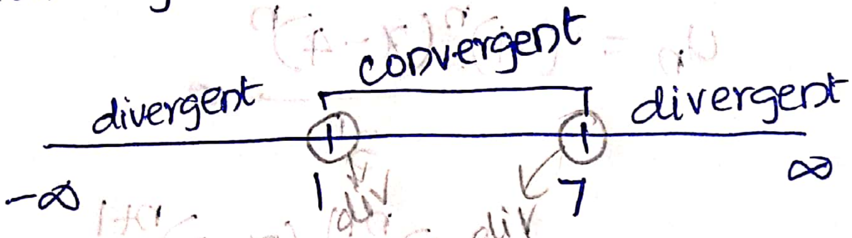
$$\therefore \rho = \frac{|x-4|}{3}$$

If $\rho < 1$, the given series is convergent.

If $\rho < 1 \Rightarrow \frac{|x-4|}{3} < 1 \Rightarrow |x-4| < 3 \Rightarrow -3 < x-4 < 3$

$\Rightarrow 1 < x < 7$

$\rho < 1 \Rightarrow 1 < x < 7$ for which the given series is convergent.



If $\rho > 1$, the given series become divergent.

$$\frac{|x-4|}{3} > 1 \Rightarrow |x-4| > 3$$

$$\Rightarrow x-4 < -3 \text{ or } x-4 > 3$$

$$\Rightarrow x < 1 \text{ or } x > 7$$

At $x=1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{3^n}{3^n} = \sum_{n=0}^{\infty} 1$$

$\therefore$ The given series is divergent at $x=1$.

i.e, $x=1$ is a point of divergence.

At $x=7$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n = 1-1+1-\cdots$$

∴ The given series is divergent and $x=7$ is a point of divergence.

∴ Region of convergence is $(1,7)$.

Radius of convergence is 3.

$7-1=6$

6 - diam

$r=\frac{6}{2}$

$=3$